Engineering Notes

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Estimation of the Lower Stability Limit of Locally Buckled Spherical Shells

M. S. El Naschie*
Riyadh University, Riyadh, Saudi Arabia

Nomenclature

$s, \bar{s}$	= orthogonal surface coordinates
$\varphi, \bar{\varphi}$	= angular coordinates
$b_{\alpha\beta}$	= second fundamental tensor
$R_{\iota\alpha\beta\gamma}$	= Riemann curvature tensor
$\mathring{\gamma}_{ij}$	= Lagrangian strain tensor of the reference surface
ψ_{ij}	= tensor of rotation
$(\dot{\ })$	$= \dfrac{d(\ )}{d\varphi}$
$(\ ')$	$= \dfrac{d(\ )}{d\bar{s}}$
w	= radial displacement component
v	= tangential displacement component
r	= shell radius
t	= shell thickness
E	= modulus of elasticity
ν	= Poisson's ratio
P	= external pressure
$V(\psi_{ij}; \mathring{\gamma}_{ij}; P)$	= energy functional
$\mathfrak{F}(\psi_{ij}; P)$	= transformed isometric functional
$b; \bar{b}$	= wave length of the isometric transformation
$\mathcal{K}_1;\mathcal{K}_2;\bar{\mathcal{K}}_1;\bar{\mathcal{K}}_2$	= Pogorelov-Kirste coefficients
i	= damping parameter

Introduction

BESIDES the cylindrical shell under axial compression, the spherical shell under external pressure is one of the most notorious problems in structural stability. The discrepancy between the theoretical and experimental data for the pressurized spherical shell has lead to extensive studies of this problem. The theoretical research activity in this field can roughly be divided into two groups. The first uses the von Karman large deflection method search for a lower stability limit i.e., the minimum of the post buckling curve.[1,2] The drawback of this method was discussed in Refs. 3 and 4. The second group of Research Workers used Koiter's initial post buckling approach.[5,6] However, unlike the cylindrical shell under axial compression, the initial post buckling results for the spherical shell were somewhat disappointing as pointed out by Koiter himself. All the same, both methods involve a great deal of mathematics and algebra and cannot be described as exactly appealing to the engineer.

Received June 17, 1976; revision received Oct. 5, 1976.. A modified version of this paper was presented to the First Congress of International Technology, Pittsburgh, Pa., Oct. 3-8, 1976.
Index category: Structural Stability Analysis.
*Assistant Professor; also Research Associate, University College, London, England. Member AIAA.

In the following, we are giving an accurate estimate of the lower stability limit (Durchschlaglast) of the spherical shell under uniform external pressure. The method used is very simple and the result can virtually be obtained in a few lines of analysis. In a previous work on the lower stability limit of axially compressed cylindrical shells[7,8] we made use of the strut on an elastic foundation[4,9] in order to obtain the lower stability limit. Similarly, we will be making use of a recent work on the initial post buckling of the ring in an elastic foundation[10] in order to obtain the lower stability limit of the pressurized spherical shell. Although our method is exclusively of an approximative nature, the results obtained are in surprisingly good agreement with the experimental evidence. This seems to confirm a previous belief that although the isometric transformation of a shell surface is by no means the complete answer to shell buckling, it does include a large part of the reality. The analysis consists of two main steps. First, the reduction of the buckling problem to a one-dimensional one. Of course, this implies certain simplifications. To show that these simplifications are permissible, the results of Flügge's accurate spherical shell theory are reproduced using the simplified one-dimensional equations. The second step is to perform an isometric transformation of the governing equations which correspond to a local isometric transformation of a small part of the shell.[7,8] To check on the validity of this step the results of von Karman's nonlinear analysis are reproduced using our linear isometric method.

Finally, an attenuated form of local buckling is considered and the results were found to be in excellent agreement with the experimental ones and smaller than that obtained by von Karman, et al. and Friedrichs, thus fitting the experimental results better. However, the striking point, of course, remains that all our results are obtained from a linear analysis and in a very elementary fashion. Examples for the application of this method to various problems including orthotropic shells can be found in Refs. 7, 8, and 13-15. The idea of using linear analysis for the post buckling of a spherical shell was pioneered by Ashwell.[21]

Assumptions

A shell surface is, in general, a two-dimensional Riemann space for which the integrability conditions are given by the Mainardi-Codazzi and Gau β equations[12]

$$b_{\alpha\beta}|_\gamma = b_{\alpha\gamma}|_\beta \tag{1}$$

$$R_{\iota\alpha\beta\gamma} = b_{\alpha\beta}b_{\gamma\iota} - b_{\alpha\gamma}b_{\beta\iota} \tag{2}$$

where $b_{\alpha\beta}$ and $R_{\iota\alpha\beta\gamma}$ are the second fundamental tensor and the Riemann curvature tensor respectively. In the immediate

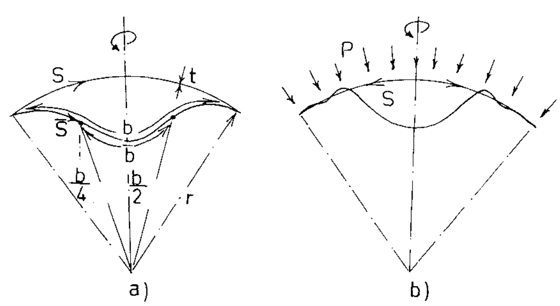

Fig. 1 The local isometric buckling of a spherical shell.

vicinity of any point on a sphere, it can easily be shown that the first equation is identically, and the second equation is approximately, satisfied. Consequently, a small part of the sphere can always be regarded as quasi-developable. Thus, the local deformation in the post buckling region will be dominated by the bending action[7,8,13,14] and the axioms of the Euclidean plane geometry are locally valid.

The Reduced Potential Energy

Following Ref. 10, the potential energy functional of a ring in an elastic foundation under external pressure can be written as

$$V=\oint\Big\{\frac{Et^3}{24(1-\nu^2)r}\dot{\psi}^2+\frac{1}{2}(Pr/Et)^2(\cos\psi)^2 r + Pr^2[\cos\psi(1-(Pr/Et)\cos\psi)] + \frac{1}{2}c\Big[\oint(\sin\psi+(Pr/Et)\sin\psi\cos\psi-v)\mathrm{d}\varphi\Big]^2\Big\}\mathrm{d}\varphi \qquad (3)$$

In addition to this we have the auxilary condition

$$\sin\psi(1+(Pr/Et)\cos\psi)r-\dot{w}-v=0 \qquad (4)$$

where t is the thickness of the ring, r is the radius, P is the external pressure, ψ is the angle of rotation, w is the radial displacement, v is the tangential displacement component, $(\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}\varphi$ and φ is the angular coordinate. We can now achieve a reduction of our spherical shell buckling problem to a one-dimensional problem using the analogy between the ring in an elastic foundation and the spherical shell by replacing EI by $Et^3/12(1-\nu^2)$, Pr by $Pr/2$ and c by Et/r^2. Equations (3) and (4) can be used to investigate the axisymmetrical buckling of a spherical shell of the same thickness and radius, t and r under external dead pressure and the result can be shown to lie within a permissible approximation. The obtained critical pressure is

$$P^c=2E(t/r)^2[\sqrt{3(1-\nu^2)}]^{-1} \qquad (5)$$

This is exactly the same value predicted by the exact equations of Flügge,[19] which verifies the validity of the above simplified formulation. As an example of an up-to-date discussion of this approximation the reader is referred to the work of Dym.[11]

The Isometric Transformation

The most important consequence of our assumptions about the quasi-developable surface is that we are justified in performing an isometric transformation of the energy functional[7,8,13]

$$V(\psi_{ij};\ \mathring{\gamma}_{ij};\ P)\rightarrow\mathfrak{F}(\psi_{ij};\ P);\ \mathring{\gamma}_{ij}=0 \qquad (6)$$

That means for a periodical or a one-way local buckling, the coefficients of the corresponding differential equation can be replaced by that of Pogorelov-Kirste[13,14] or for the $V(\psi_{ij};\ \mathring{\gamma}_{ij};\ P)$ functional

$$t^3/24(1-\nu^2)\rightarrow\tilde{\mathcal{H}}_1 \qquad t/2r^2\rightarrow\tilde{\mathcal{H}}_2 \qquad (7)$$

where

$$\tilde{\mathcal{H}}_1=tb^5/1440r^2$$

and

$$\tilde{\mathcal{H}}_2=(t\delta)^3/12(1-\nu^2)b^3$$

as discussed in Refs. 7 and 8 and b is the isometric wave length of the φ direction.

In addition to the isoperimetrical conditions

$$v=\oint w\mathrm{d}\varphi \qquad (8)$$

and

$$\psi=\mathrm{d}w/\mathrm{d}\varphi \qquad (9)$$

this leads to a very simple differential equation

$$\overset{....}{\omega}+(Pr/Et)\mathcal{H}_1\ddot{\omega}+\mathcal{H}_2\omega=0 \qquad (10)$$

where

$$\mathcal{H}_1=\frac{360\ r^2}{b^4},\quad \mathcal{H}_2=\frac{120\ \pi^3(tr)^2}{b^8(1-\nu^2)}$$

In other words, we are treating the advanced final buckling configuration of the spherical shell as a ring in a "bending" foundation. That is to say, the foundation stiffness is due to the bending rigidity of the orthogonal elements rather than the membrane hoop stresses. Since the deformation of a quasi-developable surface is essentially isometric, the membrane energy is rendered approximately zero as the shell middle surface hardly suffers any stretching. The bending stiffness of the shell forms the main resistance to deformation, at least in the advanced post buckling. This can also be shown, in a mathematically more elegant way, to be in accordance with the Gauss principle of the least constraint.[16] Recalling the known experimental fact that the shell buckling is a local phenomenon the preceding thinking pattern forms a simple and logical interpretation of shell buckling in general.[8]

Reproduction of the Results of von Karman et al.

In order to gain some confidence in the proposed method and to confirm the validity of the corresponding formulation as a reasonable approximation for the lower stability limit, we seek a duplication of the results of von Karman et al.[1] To do so, we need only take the solution

$$w=\sum_i a_i\sin i\varphi \qquad (11)$$

Inserting Eq. (11) in our potential functional, Eq. (6), a Ritz procedure yields

$$P^c=0.37\ E(t/r)^2 \qquad (12)$$

This is nearly identical to the value obtained by von Karman[1]

$$P^c_{\min}=0.366\ E(t/r)^2$$

Two-Way Local Attenuated Buckling Mode (Local Dimple)

We now come to consider the two-way local buckling (that is to say local dimple) suggested by experimental observation. To do so, we have to use a modified isometric function $\mathfrak{F}_D$ to suit the new condition. If we take, for the sake of simplicity, the function in an orthogonal direction to $w_{(s)}$, that is to say $\bar{s}$, to be

$$\int_0^b f\mathrm{d}\bar{s}=\int_0^b w_0\Big(1-\cos\frac{2\pi}{b}\bar{s}\Big)\mathrm{d}\bar{s} \qquad (13)$$

then the modified Pogorelov-Kirste coefficients corresponding to Eq. (7) can easily be obtained as follows. Isolating the middle part of the orthogonal function (15) which forms the cross-section of our fictitious curved column, the constant of the fictitious bending foundation is (see Fig. 1)

$$C_B = \frac{Et^3}{12(1-\nu^2)} \left(2\int_{b/4}^{b/2} f'''' \mathrm{d}\bar{s}\right) \frac{1}{w_0}; \quad (\,'\,) = \frac{\mathrm{d}\,(\,)}{\mathrm{d}\bar{s}}$$

$$= \frac{8}{6} \frac{Et^3}{(1-\nu^2)} \left(\frac{\pi}{b}\right)^3 \tag{14}$$

and since the wavelength of the isolated column is obviously (see Fig. 1)

$$\bar{b} = b/2 \tag{15}$$

we can replace

$$t^3/24(1-\nu^2) \rightarrow t(b/2)^5/1440\, r^2 \tag{16}$$

and

$$t/2r^2 \rightarrow 8t^3\pi^3/12(1-\nu^2)b^3$$

Inserting the following simple attenuated buckling mode

$$w = a^{-is} e(\cos is + \sin is); \quad s = \varphi r$$

in the corresponding differential equation, a simple Galerkin procedure leads to

$$P^c = ½(0.65\, Et^2/r^2) \cong 0.325\, E(t/r)^2 \tag{17}$$

Comparing the critical value obtained above with some of the experimental results, we find they are in good agreement with those reported by Fung and Sechler.[17] The most interesting point is that this value lies much lower than the value obtained by Friedrichs $P^c_{\min} = 0.9\,E(t/r)^2$ who has considered a transition condition from the buckled to the unbuckled region of the spherical shell which is somewhat similar to that corresponding to Eq. (19).

Acknowledgment

This research was conducted as a part of research project No. AM/P11 which is supported by a grant from the University of Riyadh. The author wishes to express his thanks to his colleagues at the Faculty of Engineering for the pleasant research atmosphere provided.

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Technical Comments

Comment on "Optimum Low-Thrust Rendezvous"

T.N. Edelbaum*
Charles Stark Draper Laboratory, Inc., Cambridge, Mass.

REFERENCE 1 contains two linearized solutions for optimum low-thrust orbital rendezvous with ideal power-limited rocket engines. The solution for circular orbits is equivalent to the one obtained by Gobetz[2] more than a decade ago. The solution for elliptical orbits is restricted to small changes in radius and true anomaly. It is thus limited to short duration maneuvers and is somewhat incompatible with the low-thrust assumption which implies long maneuver duration. The author also claims that his elliptical orbit solution is valid for small eccentricity and longer duration. However, it is probably less accurate for these cases than his circular orbit solution.

The apparent reason for the additional restrictive assumption in the elliptical case is the desire to reduce the problem to a constant coefficient linear system, which is integrable. If the problem is linearized around a target in an elliptic orbit, it requires integration of a time varying linear system. This is apparently why the author of Ref. 1 introduced his additional assumptions. However, by using eccentric anomaly as the independent variable, it becomes possible to integrate this time varying linear system in terms of elementary functions.[3] The resulting solution is valid for arbitrary eccentricity and duration within the linear approximation.

In addition, by using a particular set of orbital elements as variables, additional results have been obtained for the practically interesting case of long duration maneuvers. In this latter case the optimal thrust programs become orthogonal, so that the optimal thrust program for each element produces no

Received Aug. 5, 1976
Index category: Earth-Orbital Trajectories.
*Staff Member. Associate Fellow AIAA.